

Notes on Hidden Markov Model

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1 Hidden Markov Model

Hidden Markov Model (HMM) is a parameterized distribution for sequences of observations. An intuitive way to explain HMM is to go through an example. Suppose that Ken hears (a.k.a. observes) a sequence of T sounds o_1, o_2, \dots, o_T and he wants to reason something about this sequence. He makes the assumption that the sequence of sounds that he heard depends on a sequence of T words s_1, s_2, \dots, s_T , which he never gets to see and which is why they are called the hidden states. HMM gives Ken a method which, under certain assumptions, allows him to assign appropriate probabilities to sound sequences O 's and word sequences S 's and to make reasonable deductions about them. Figure 1 illustrates an HMM with $T = 5$.

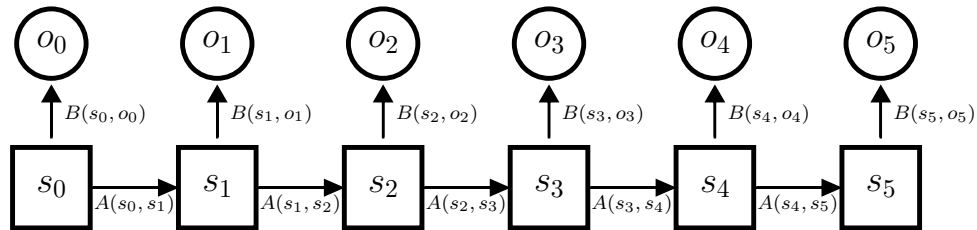


Figure 1: An illustration of a Hidden Markov Model with 5 steps

HMM makes several assumptions about the data it models:

1. The observations o_1, o_2, \dots, o_T come from a known, finite sets V , called the observation space.
2. The hidden states s_1, s_2, \dots, s_T come from a known, finite sets Q , called the hidden state space.
3. The HMM assigns a probability to any given sequence s_1, s_2, \dots, s_T . The chain rule says that

$$P(s_1, s_2, \dots, s_T) = \prod_{t=1}^T P(s_t | s_0, s_1, \dots, s_{t-1}) = \prod_{t=1}^T P(s_t | s_{<t}) \quad (1.1)$$

Here, we are being a little sloppy in terms of notation by assuming that there is a starting state s_0 , just so that the indexing is valid. In practice, such state s_0 is sometimes called the starting state, and one can assign a probability distribution to it too.

HMM goes further than Equation 1.1 by making the assumption that

$$P(s_t | s_{<t}) = P(s_t | s_{t-1}) \quad (1.2)$$

This is called the **Markov assumption**. With this assumption, Equation 1.1 can be rewritten as

$$P(s_1, s_2, \dots, s_T) = \prod_{t=1}^T P(s_t | s_{t-1}) \quad (1.3)$$

Note that the Markov assumption generally does not hold. In our example, sequences of words usually have dependencies that go back longer than one immediate step.

4. Each observation o_t only depends on the immediate hidden state s_t . Formally

$$P(o_t | o_{<t}, s_{<t}) = P(o_t | s_t) \quad (1.4)$$

This is called the **independence assumption**.

Under the Markov assumption and the independence assumption, an HMM only needs two set of parameters to model $P(s_t | s_{t-1})$ and $P(o_t | s_t)$. We call these, respectively, the *transition probabilities* and the *emission probabilities*. These probabilities are denoted by matrices A and B , each of whose row is a valid probability distribution, i.e. non-negative numbers that sum up to 1. Table 1 summarizes the components of a general HMM.

Name	Notation	Meaning/Property
State space	$Q = \{q_1, q_2, \dots, q_N\}$	Set of N states
Observation space	$V = \{w_1, w_2, \dots, w_V\}$	Set of V states
State sequence	$S = \{s_1, s_2, \dots, s_T\}$	Sequence of T steps. $s_i \in Q$
Observation sequence	$O = \{o_1, o_2, \dots, o_T\}$	Sequence of T steps. $o_i \in V$
Transition probs	$A \in \mathbb{R}^{N \times N}$	Each row is a valid distribution
Emission probs	$B \in \mathbb{R}^{N \times V}$	Each row is a valid distribution

Table 1: Components of a Hidden Markov Model.

There are three important questions regarding HMM.

- (1) **Computing the Likelihood.** Given A, B and a sequence of observations O . Compute $P(O|A, B)$.
- (2) **Finding the Hidden Sequence.** Given A, B and a sequence of observation O . Find the hidden sequence S that is most likely to generate O , that is to find

$$S^* = \operatorname{argmax}_S P(O|S, A, B) \quad (1.5)$$

- (3) **Estimating the Parameters.** Given a sequence of observation O . Estimate the transition and emission probabilities that are most likely to give O . That is to find

$$A^*, B^* = \operatorname{argmax}_{A, B} P(O|A, B) \quad (1.6)$$

In the following sections, we provide the algorithms to address each of them.

2 Computing the Likelihood: Forward-Pass Algorithm

Given a sequence $O = (o_1, o_2, \dots, o_T)$ and known transition and emission probabilities matrices A, B . We want to find $P(O|A, B)$. We use dynamic programming. Specifically, for each step $t = 1, 2, \dots, T$ and each hidden state s , we will compute and cache the following value

$$f[t, s] = P(o_1, o_2, \dots, o_t, s_t = s | A, B) \quad (2.1)$$

If we know $f[t, s]$, we can compute $P(O|A, B)$ as follows

$$P(o_1, o_2, \dots, o_t | A, B) = \sum_s P(o_1, o_2, \dots, o_t, s_t = s | A, B) \cdot P(o_t | s_t = s) \quad (2.2)$$

$$= \sum_s f[t, s] \cdot B(s, o_t) \quad (2.3)$$

Now we can compute $f[t, s]$ from previous values $f[t-1, s']$. Specifically

$$f[t, s] = P(o_1, o_2, \dots, o_t, s_t = s | A, B) \quad (2.4)$$

$$= \sum_{s'} P(o_1, o_2, \dots, o_{t-1}, s_{t-1} = s' | A, B) \cdot P(s_t = s | s_{t-1} = s') P(o_t | s) \quad (2.5)$$

$$= \sum_{s'} f[t-1, s'] \cdot A(s', s) B(s, o_t) \quad (2.6)$$

Putting together Equation 2.3 and Equation 2.6, we have the so-called *Forward-Pass* algorithm to compute the likelihood in an HMM.

(1) Initialize:

- For each hidden state s :

$$f[1, s] = P(o_1, s_1 = s | A, B) \leftarrow B(s, o_1) \cdot A(s_0, s) \quad (2.7)$$

(2) For $t = 2$ to T :

- For each hidden state s :

$$f[t, s] \leftarrow \sum_{s'} f[t-1, s'] \cdot A(s', s) B(s, o_t) \quad (2.8)$$

(3) Finally:

$$P(o_1, o_2, \dots, o_T | A, B) \leftarrow \sum_s f[T, s] B(s, o_T) \quad (2.9)$$

Its complexity is $O(N^2 \cdot T)$ where N is the size of the hidden state space.

3 Finding the Hidden Sequence: Viterbi Algorithm

Given a sequence $O = (o_1, o_2, \dots, o_T)$ and known transition and emission probabilities matrices A, B . For each hidden sequence S , the joint probability that both O and S happen is

$$P(O, S | A, B) = P(O | S, A, B) \cdot P(S | A, B) \quad (3.1)$$

$$= \prod_{i=1}^T P(o_i | s_i) \cdot \prod_{i=1}^T P(s_i | s_{i-1}) \quad (3.2)$$

$$= \prod_{i=1}^T B(s_i, o_i) \cdot \prod_{i=1}^T A(s_{i-1}, s_i) \quad (3.3)$$

We want to find the sequence S that maximizes the quantity above, that is

$$S^* = \operatorname{argmax}_S P(O, S|A, B) \quad (3.4)$$

It turns out that a dynamic programming programming algorithm that is almost identical to the Forward-Pass algorithm in Section 2 is applicable for the task of finding S in Equation 3.4. That is the Viterbi algorithm [Viterbi, 1967].

We define

$$g[t, s] \stackrel{\text{def}}{=} \max_{s_1 \dots s_{t-1}} P(o_1, o_2, \dots, o_t, s_t = s|A, B) \quad (3.5)$$

The key observation is that if we know all values of $g[t, s]$, then we can compute the maximum joint probability $P(O, S|A, B)$ as follows

$$\max_{s_1, \dots, s_t} P(o_1, o_2, \dots, o_t|A, B) = \max_s P(o_1, o_2, \dots, o_t, s_t = s|A, B) \cdot P(o_t|s_t = s) \quad (3.6)$$

$$= \max_s g[t, s] \cdot B(s, o_t) \quad (3.7)$$

Please note the similarity between Equations 3.6-3.7 to Equations 2.2-2.3, where the only difference is the change from summation to max. The reason that Viterbi's algorithm is almost identical to Forward-Pass algorithm is that max has the same distributing property as summation, i.e. one can "factor" our the same term.

We can compute $g[t, s]$ similar to the way we compute $f[t, s]$ in Equation 2.6. Specifically

$$g[t, s] = \max_{s_1, \dots, s_t} P(o_1, o_2, \dots, o_{t-1}, o_t, s_t = s|A, B) \quad (3.8)$$

$$= \max_{s'} P(o_1, o_2, \dots, o_t, s_{t-1} = s'|A, B) \cdot P(s_t = s|s_{t-1} = s')P(o_t|s) \quad (3.9)$$

$$= \max_{s'} g[t-1, s'] \cdot A(s', s)B(s, o_t) \quad (3.10)$$

Finally, note that in Equation 3.4, we wanted an $\operatorname{argmax}_S P(O, S|A, B)$, not the $\max_S P(O, S|A, B)$. The general idea to go from $\max(\cdot)$ to $\operatorname{argmax}(\cdot)$ in a dynamic programming algorithm is to create a trace-back array. In Equation 3.10, whenever you know which s' leads to the largest value for $g[t-1, s'] \cdot A(s', s)B(s, o_t)$, you can store that s' in an array $h[t, s]$. After you finish with $g[T, s]$ as in Equation 3.7, you can simply follow $h[t, s]$ to trace back the sequence of hidden states that leads to the maximum value of $P(O, S|A, B)$, which gives you the argmax . Putting everything together, we have the Viterbi's algorithm

(1) Initialize:

- For each hidden state s :

$$g[1, s] \leftarrow B(s, o_1) \cdot A(s_0, s) \quad (3.11)$$

(2) For $t = 2$ to T :

- For each hidden state s :

$$g[t, s] \leftarrow \max_{s'} g[t-1, s'] \cdot A(s', s)B(s, o_t) \quad (3.12)$$

$$h[t, s] \leftarrow \operatorname{argmax}_{s'} g[t-1, s'] \cdot A(s', s)B(s, o_t) \quad (3.13)$$

(3) Follow $h[t, s]$ to find $s_T^*, s_{T-1}^*, \dots, s_1^*$. Starting at $t = T$

$$s_T^* \leftarrow \underset{s}{\operatorname{argmax}} g[T, s] \quad (3.14)$$

$$s_t^* \leftarrow h[t+1, s_{t+1}^*] \quad \text{for } t = T-1, T-2, \dots, 1 \quad (3.15)$$

Viterbi's algorithm also has the complexity of $O(N^2 \cdot T)$.

4 Estimating the Parameters: Baum-Welch Algorithm

In Section 2 and Section 3, we have seen that if we know the transition probabilities $A(s, s')$ and the emission probabilities $B(s, o)$, then we can efficiently perform the queries on an HMM. In this section, we are concerned with the problem of estimating A and B , given a single sequence of observations $O = (o_1, o_2, \dots, o_T)$.

The algorithm for this task is the Baum-Welch algorithm [Baum, 1972], an instance of a procedure called Expectation-Maximization (EM) [Dempster et al., 1977]. In Baum-Welch algorithm, one starts by randomly initializing the values for A and B , then repeatedly updating them. Each updating iteration has two steps, the Expectation step (E-step) and the Maximization step (M-step).

E-step. The E-step assumes that we know A, B and computes the following quantities

$$\gamma[t, s] = P(s_t = s | O, A, B) \quad (4.1)$$

$$\xi[t, s', s] = P(s_{t-1} = s', s_t = s | O, A, B) \quad (4.2)$$

Intuitively, $\gamma[t, s]$ counts how many times does the t^{th} states of the hidden sequence equals s and $\xi[t, s, s']$ counts how many times the duo (s', s) happens at the $(t-1)^{\text{th}}$ step and the t^{th} step in the hidden sequence, both up to normalization by a partition function. That is why $\gamma[t, s]$ and $\xi[t, s, s']$ are sometimes referred to as the *pseudo counts*.

We now show how to compute $\gamma[t, s]$ and $\xi[t, s', s]$. Since $\gamma[t, s]$ and $\xi[t, s', s]$ are probabilities conditioned on O , we compute via the partial joint probabilities

$$\alpha[t, s] = P(o_1, o_2, \dots, o_t, s_t = s | A, B) \quad (4.3)$$

$$\beta[t, s] = P(o_{t+1}, o_{t+2}, \dots, o_T | s_t = s, A, B) \quad (4.4)$$

If one knows all $\alpha[t, s]$ and $\beta[t, s]$, then $\gamma[t, s]$ and $\xi[t, s', s]$ can be computed as follows

$$\gamma[t, s] = P(s_t = s | O, A, B) = \frac{P(s_t = s, O | A, B)}{P(O | A, B)} \quad (4.5)$$

$$= \frac{\alpha[t, s] \cdot \beta[t, s]}{\sum_{s'} \alpha[t, s'] \cdot \beta[t, s']} \quad (4.6)$$

$$\xi[t, s', s] = P(s_{t-1} = s', s_t = s | O, A, B) = \frac{P(s_{t-1} = s', s_t = s, O | A, B)}{P(O | A, B)} \quad (4.7)$$

$$= \frac{\alpha[t-1, s'] \cdot A(s', s) \cdot \beta[t, s]}{\sum_{s'} \alpha[t, s'] \cdot \beta[t, s']} \quad (4.8)$$

It only remains how to compute $\alpha[t, s]$ and $\beta[t, s]$. Not surprisingly, they can be computed with dynamic programming

$$\alpha[t, s] = \sum_{s'} \alpha[t-1, s'] \cdot A(s', s) \cdot B(s, o_t) = B(s, o_t) \sum_{s'} \alpha[t-1, s'] \cdot A(s', s) \quad (4.9)$$

$$\beta[t, s] = \sum_{s'} \beta[t + 1, s'] \cdot A(s, s') \cdot B(s', o_{t+1}) \quad (4.10)$$

M-step. When these pseudo counts γ and ξ are known, then in the M-step, one can use maximum likelihood estimator to derive the updated values for A and B

$$\hat{A}(s', s) = \frac{\text{\#times } s \text{ follows } s'}{\text{\#times anything follows } s'} = \frac{\sum_{t=1}^{T-1} \xi[t, s', s]}{\sum_{s''} \sum_{t=1}^{T-1} \xi[t, s', s'']} \quad (4.11)$$

$$\hat{B}(s, o) = \frac{\text{\#times } o \text{ is observed given } s}{\text{\#times anything is observed given } s} = \frac{\sum_{t=1}^T \mathbf{1}[o_t = o] \gamma[t, s]}{\sum_{t=1}^T \gamma[t, s]} \quad (4.12)$$

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